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# On Generalized Quantum Deformations and Symmetries in Quantum Mechanics

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## Abstract

Supersymmetric and parasupersymmetric quantum mechanics are now recognized as two further parts of quantum mechanics containing a lot of new informations enlightening (solvable) physical applications. Both contents are here analysed in connection with generalized quantum deformations. In fact, the parasupersymmetric context is visited when the order of paraquantization  $p$  is limited to the first nontrivial value  $p = 2$ .

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# 1 Introduction

Recent developments in quantum mechanics are due to simple ideas related to the superposition of different types of particles described by specific statistics. The simplest context surely is that of supersymmetric quantum mechanics (SSQM) subtending the idea of superposition of usual (integer spin) bosons satisfying the very well known Bose-Einstein statistics with usual (half integer spin) fermions satisfying the very well known Fermi-Dirac statistics. Such an idea, first developed by Witten [1], is evidently issued from the rich concept of supersymmetry initially exploited in particle physics in the seventies [2]. Its recent developments in SSQM are quoted in two review papers [3].

Another context, intimately connected with the previous one, is that of parasupersymmetric quantum mechanics (PSSQM) essentially developed at the start by Rubakov-Spiridonov [4] and Beckers-Debergh [5], this field subtending now the idea of superposition of (once again) usual bosons but with parafermions, i.e. “particles” characterized by specific trilinear structure relations associated with parastatistics of order  $p$  [6, 7]. Let me only recall that, for  $p = 1$ , the parafermionic context reduces to the above fermionic one, such a remark justifying that, in the following PSSQM developments, I want to consider  $p \geq 2$ . PSSQM has, in particular, specific interests in the study of interactions for real particles (i.e. when  $p = 2$ , nonzero rest mass vector mesons in external (electro)magnetic fields [8] have been studied).

Both contexts can then be visited in connection with recent developments in quantum deformations [9] and, more particularly, in generalized quantum deformations according to specific points of view [10]. Let me say that, at the levels of  $q$ -deformed SSQM and PSSQM, there are already published works [11, 12] which will not enter into the present discussion. I mainly want to insist here on the interest of generalized quantum deformations leading to recent results in SSQM as well as in PSSQM. The contents of this communication are then distributed as follows. In Section 2, I recall a few necessary ingredients and, in Sections 3 and 4, I successively consider the SSQM- and PSSQM-contexts and some of their generalized deformations. In the parasupersymmetric case, I limit myself to the  $p3D2$ -study while the general one for arbitrary  $p$ 's is already available [13].

## 2 A few necessary ingredients

For evident reasons, it is not possible to insert here the complete set of ingredients covering SSQM- and PSSQM-characteristics, but I can mention useful references for necessary information besides a few relations.

Let me recall that  $N = 2$ -SSQM is, after Witten [1], characterized by a Lie superalgebra [14] called  $\text{sqm}(2)$  generated by two supercharges  $Q_1$  and  $Q_2$  (or  $Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$  and  $Q^+$ ) such that

$$\{Q, Q^+\} = H_{SS}, \quad [Q, H_{SS}] = 0, \quad [Q^+, H_{SS}] = 0, \quad Q^2 = Q^{+2} = 0 \quad (1)$$

where  $H_{SS}$  appears as the supersymmetric Hamiltonian made of a bosonic as well as a fermionic parts, the supercharges being expressed in terms of a superpotential  $W(x)$  (if I consider and limit myself to 1-dimensional problems). It is not possible to  $q$ -deform such a structure [11] in contradistinction with Spiridonov's claim, but it is possible [15] through generalized deformations as shown in Section 3. An important property entering in such a construction is to learn that SSQM is also realized through the superposition of parabosons and parafermions of all orders  $p$  ( $\neq 1$ ) if we combine necessarily pairs of the same orders [16]. Such a property is obtained through Green-Cusson Ansatze [17], i.e. when the realizations of  $p = 2$ -parabosonic ( $a$ ) and parafermionic ( $b$ ) annihilation operators are

$$a = \sum_{\alpha=1}^{p=2} A_\alpha \xi_\alpha, \quad b = \sum_{\beta=1}^2 B_\beta \xi_\beta \quad (2)$$

where

$$[A_\alpha, A_\beta^+] = \delta_{\alpha\beta}, \quad \{B_\alpha, B_\beta^+\} = \delta_{\alpha\beta}, \quad \{\xi_\alpha, \xi_\beta\} = 2\delta_{\alpha\beta}, \quad (3)$$

referring to usual bosonic ( $A_\alpha$ ,  $\alpha = 1, 2$ ) and fermionic ( $B_\alpha$ ,  $\alpha = 1, 2$ ) operators while the  $\xi$ 's generate  $\mathcal{C}\ell_2$ -Clifford algebras. Discussions on Fock bases associated with operators such as (2) can be presented following Macfarlane's developments [18], so that we can enter two usual bosons ( $a_1, a_2$ ) and four usual fermions ( $b_1, b_2, b_3, f$ ) with a view to study the action of the eight  $osp(2|2)$ -symmetry operators on this space (see Refs. [15] and [19]). One gets, in correspondence with eqs.(2), that

$$a = \sqrt{2}(a_1 f + a_2 f^+), \quad b = \sqrt{2}[b_1(b_3 + b_3^+)f + i b_2(b_3 - b_3^+)f^+] \quad (4)$$

with

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{2}}(A_1 + iA_2), & a_2 &= \frac{1}{\sqrt{2}}(A_1 - iA_2), & [a_1, a_2] &= 0, & [a_\alpha, f] &= 0, \\ \{f, f^+\} &= 1, & [b_1, b_2] &= \dots = 0, & [b_j, f] &= 0, & j &= 1, 2, 3. \end{aligned}$$

The corresponding  $osp(2|2)$ -operators [20] given by

$$\begin{aligned} H_{PB} &= \frac{1}{2}\{a, a^+\}, & H_{PF} &= \frac{1}{2}[b, b^+], & C_+ &= \frac{1}{2}\{a, a^+\}, & C_- &= \frac{1}{2}\{a, a\}, \\ Q &= \frac{1}{2}\{a, b\}, & Q^+ &= \frac{1}{2}\{b^+, a^+\}, & S &= \frac{1}{2}\{b^+, a\}, & S^+ &= \frac{1}{2}\{a^+, b\} \end{aligned} \quad (5)$$

satisfy the trilinear structure relations of the “relative parabosonic set” [7] whose specific characteristics are

$$\begin{aligned} [\{a, b\}, a^+] &= -[\{a^+, b\}, a] = 2b, \\ \{\{a, b^+\}, b\} &= \{\{a, b\}, b^+\} = 2a, \\ [\{a, b\}, a] &= [\{a, b\}, b] = 0, \\ [\{a^+, b\}, a^+] &= \{\{a, b^+\}, b^+\} = 0, \end{aligned} \quad (6)$$

besides common ones (with the "relative parafermionic set" [7]). These supersymmetric developments [16] when the same order(s) of paraquantization is (are) considered lead to three  $osp(2|2)$ -irreducible unitary representations (a typical and two atypical ones) which open the way to succeed in deforming SSQM (cf. Section 3).

As a last ingredient, let me introduce the new structure subtended by PSSQM when the Beckers-Debergh approach [5] is taken as the starting point. In  $N = 2$ -PSSQM and in generalization with respect to SSQM, we are concerned with Lie parasuperalgebras instead of Lie superalgebras. We have introduced [5] double commutation relations, so that the fundamental  $psqm$  (2) is characterized by the following structure relations

$$\begin{aligned} [Q, [Q^+, Q]] &= QH_{PSS}, & [Q^+, [Q, Q^+]] &= Q^+H_{PSS}, \\ [Q, H_{PSS}] &= [Q^+, H_{PSS}] = 0, & Q^3 &= Q^{+3} = 0, \end{aligned} \quad (7)$$

where  $H_{PSS}$  appears as the parasupersymmetric Hamiltonian made of a bosonic as well as a  $p = 2$ -parafermionic parts, the parasupercharges  $Q$  and  $Q^+$  being now expressed in terms of two superpotentials  $W_1(x)$  and  $W_2(x)$ . The latter are constrained by the relation

$$W_2^2(x) + W'_2(x) = W_1^2(x) - W'_1(x) + c_1 \quad (8)$$

where primes refer to spatial derivatives as usual.

### 3 SSQM and generalized deformations

Let me discuss very briefly two kinds of generalized deformations, each of them having specific properties in connection with SSQM.

#### 3.1 Possible deformations of SSQM

As already proposed elsewhere [15, 19], we follow the Quesne suggestion [10] leading to the substitution of our parabosonic and parafermionic operators by new deformed ones as follows:

$$a \longrightarrow A = \frac{1}{2\sqrt{2}} (\sqrt{F_2(N)} a^2 a^+ - \sqrt{F_1(N)} a^+ a^2) \quad (9)$$

and

$$b \longrightarrow B = \frac{1}{2\sqrt{2}} (\sqrt{F_1} b^+ b^2 + \sqrt{F_2} b^2 b^+), \quad (10)$$

where, in particular,  $N$  is the number operator and  $F_1, F_2$  some arbitrary functions. Through the construction of the operators (4) and (5) as well as through their actions on the (unchanged) Fock basis entering two bosons and four fermions, it is possible [15] to see that the corresponding typical irreducible representation of  $osp(2|2)$  shows that it contains a deformed  $sqm$  (2)-representation while the other two atypical ones are not deformed. Some nilpotencies are now of the third order instead of the second one as expected in  $sqm$  (2).

## 3.2 Reducibility of SSQM

Here I suggest the substitution

$$a \longrightarrow A = \frac{1}{2\sqrt{2}} a (1 + P) \quad (11)$$

at the level of the bosonic annihilation operator and, consequently,

$$a^+ \longrightarrow A^+ = \frac{1}{2\sqrt{2}} a^+ (1 - P) \quad (12)$$

at the level of the corresponding creation operator, where  $P$  is the parity operator admitting  $(-1)^n$  (for all integers  $n$ ) as eigenvalues. In fact, these are once again deformed operators in the sense that, acting on a Fock basis  $\{|n\rangle\}$ , we get

$$A |n\rangle = \sqrt{F(n)} |n-1\rangle, \quad A^+ |n\rangle = \sqrt{F(n+1)} |n+1\rangle \quad (13)$$

with

$$\sqrt{F(n)} = \frac{1}{\sqrt{2}}(1 + (-1)^n)\sqrt{n}, \quad \sqrt{F(n+1)} = \frac{1}{\sqrt{2}}(1 - (-1)^n)\sqrt{n+1}. \quad (14)$$

Such an approach is a certain generalization of the usual  $q$ -deformation introduced [21], for example, in the harmonic oscillator context.

The remarkable fact here is that these operators  $A$  and  $A^+$  can play the role of “supercharges” in SSQM. Indeed they generate the structure relations (1) with

$$H_{SS} = \frac{1}{2}\{A, A^+\} = \frac{1}{2}\{a, a^+\} = -\frac{1}{2}P \quad (15)$$

where  $(-\frac{1}{2}P)$  with the eigenvalues  $(\mp\frac{1}{2})$  plays the role of the fermionic Hamiltonian. Such considerations lead to exact supersymmetry [22] and to an (unexpected) reducibility of SSQM when the superpotential characteristic of the interaction is odd (but not when it is even). Specific dynamical symmetries can also be displayed [22] but cannot be discussed here : the harmonic (super)oscillator enters in the odd context while the hydrogen atom (and its superCoulomblike interaction) belongs to the even case.

## 4 PSSQM and generalized deformations

Let me come back on the superposition of a usual boson and a generalized deformed parafermion of order  $p = 2$ . The corresponding generalized deformed parafermionic operators (called here  $b$  and  $b^+$ ) transform under a 3-dimensional unitary and irreducible representation of a Polychronakos-Roček deformed  $su(2)$ -algebra [10]. From the parastatistical point of view, we are asking for generalized deformed parafermionic operators satisfying the nilpotency and trilinear relations

$$b^3 = (b^+)^3 = 0, \quad [b, [b^+, b]] = G(N) b,$$

$$[b^+, [b, b^+]] = b^+ G(N), \quad G(N) = 2F(N+1) - F(N) - F(N+2), \quad (16)$$

$$F(N) = b^+ b,$$

$F$  being any positive analytic function. We have constructed [13] associated parasupercharges leading to a deformed parasuperalgebra containing explicitly the new parasuperhamiltonian in correspondence with the structure [7]. Through the constraints (8) on the superpotentials, we can determine the diagonal parasupersymmetric elements of  $H_{PSS}$  as given by

$$H_{kk} = \frac{1}{2}p^2 + f_k(x), \quad k = 1, 2, 3, \quad (17)$$

leading to only three different Hamiltonians  $H^{(1)}$ ,  $H^{(2)}$  and  $H^{(3)}$ . It has to be noticed that  $H^{(1)}$  and  $H^{(2)}$  appear as of the  $\Xi$ -type in the Semenov-Chumakov scheme [23] while  $H^{(3)}$  is of the  $V$ -type. These specific results correspond to 3-level systems of special physical interest in quantum optics [24].

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